

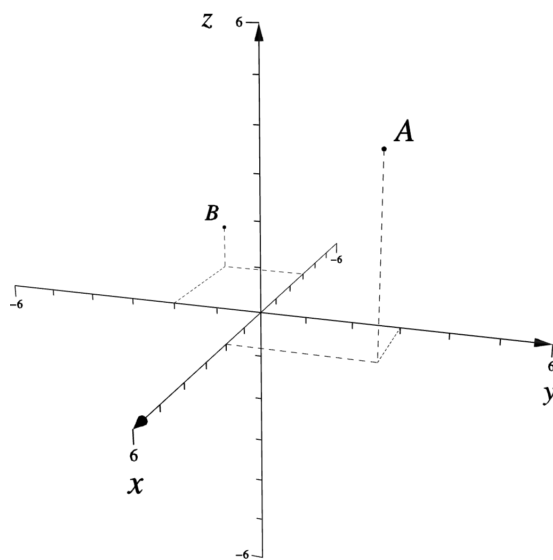
New Topics for Junior High Mathematics (2018-2019)

Spatial Algebra and Geometry ¹

Just as an equation with x and y determines a relation between algebra and geometry in a plane, an equation with x , y , and z provides a link between algebra and geometry in space.

Graphing in Space

Points created from ordered triples (x, y, z) can be graphed in space. Notice the order of the axes (pronounced *AK-seez*) on the graph. The positive x -axis comes forward, the positive y -axis goes to the right, and the positive z -axis goes up. The points $A(2, 3, 4)$ and $B(-3, -2, 1)$ have been graphed. To graph A , start at the origin and move 2 units on the x -axis in the positive direction. From this point, move to the right 3 units, parallel to the y -axis. Finally, move 4 units straight up, parallel to the z -axis. For more help in graphing, you can also graph 2 units forward parallel to the x -axis from the point $(0, 3, 0)$ along the y -axis. This will create a rectangle in the xy -plane from which you can move up, parallel to the z -axis.



Distance Formula in Space

The distance between two points in space is given by the formula

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2},$$

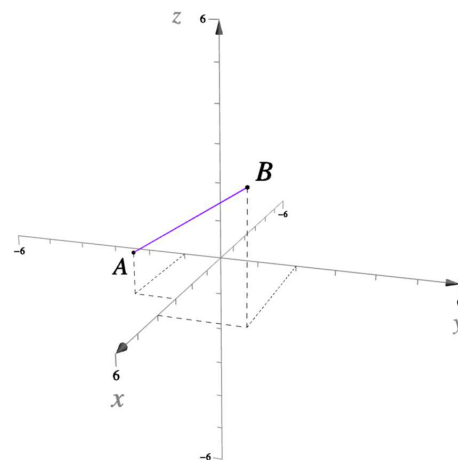
where one point is given by coordinates (x_1, y_1, z_1) and the other point is (x_2, y_2, z_2) .

EXAMPLE:

Find the distance between the points $(3, -1, 1)$ and $(4, 2, 3)$.

You can use the Distance Formula to find the distance.

$$\begin{aligned} D &= \sqrt{(4 - 3)^2 + (2 - (-1))^2 + (3 - 1)^2} \\ &= \sqrt{1^2 + 3^2 + 2^2} \\ &= \sqrt{14} \end{aligned}$$



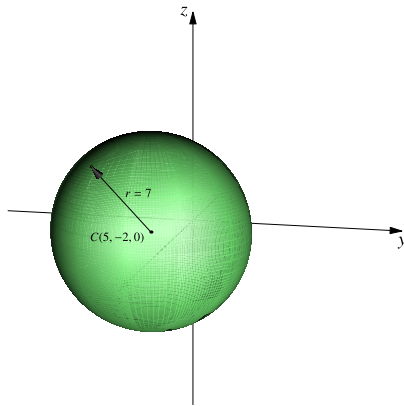
¹This document was prepared by Doug Ray for competition in years 2018 and 2019. If you have any questions about the material presented, please email doug@academicmeet.com.

Spheres

Using the Distance Formula, you can find the equation of a sphere by letting the distance equal the radius and the second point be the center of the sphere:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2,$$

where r is the radius of the sphere centered at the point (x_0, y_0, z_0) .



EXAMPLE:

Find the center and radius of the sphere whose equation is

$$(x - 5)^2 + (y + 2)^2 + z^2 = 49.$$

First, the radius is the square root of the number on the right: $r^2 = 49 \Rightarrow r = 7$.

Next, we need to match up the equation to the standard form of the sphere: $(x - 5)^2 + (y - (-2))^2 + (z - 0)^2 = 49$. Notice that the y term has been changed so that it starts as “ $y-$ ” and the z term now has a subtracted term of zero. Thus, the center is $C(5, -2, 0)$.

Planes

The general equation of a plane is $Ax + By + Cz = D$. Notice that the powers on all the variables are 1. One way to graph the plane is by finding the intercepts. The intercepts are the points where the graph passes through the axes. To find one of the intercepts, set the other variables equal to 0 and solve for that variable.

EXAMPLE:

Find the intercepts of the plane $4x - 5y + 6z = 12$.

To find the x -intercept, set y and z equal to zero and solve for x .

$$\begin{aligned} 4x - 5(0) + 6(0) &= 12 \\ 4x &= 12 \\ x &= 3 \end{aligned}$$

Therefore, the x -intercept is $(3, 0, 0)$. You can find the other intercepts in a similar way to get $(0, -\frac{12}{5}, 0)$ and $(0, 0, 2)$ for the y - and z -intercepts, respectively. Graphing these points and then forming the little triangle between them is one way of representing the plane $4x - 5y + 6z = 12$.

Volume and Surface Area

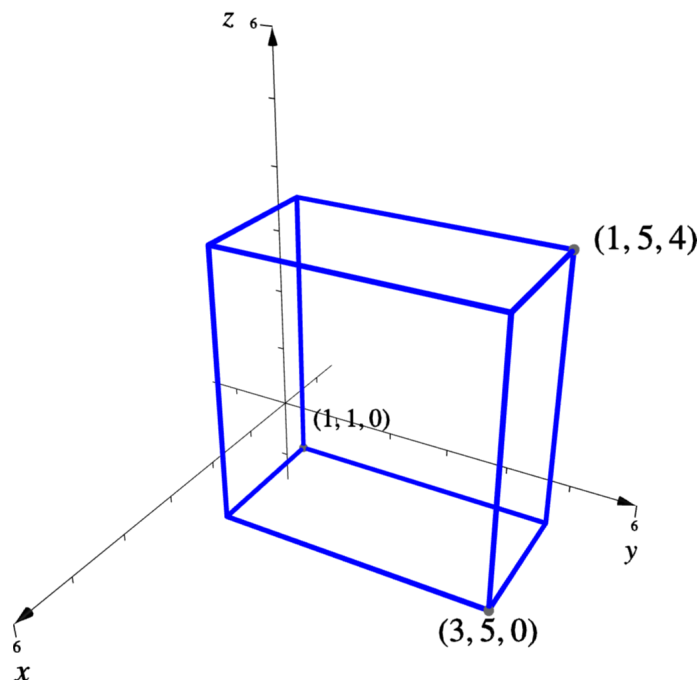
Given information about a 3D object, you can find the volume and surface area using traditional geometric formulas:

Shape	Volume	Surface Area
Sphere	$V = \frac{4}{3}\pi r^3$	$SA = 4\pi r^2$
Box	$V = \ell wh$	$TSA = 2(\ell w + wh + h\ell)$
Cone	$V = \frac{1}{3}\pi r^2 h$	$LSA = \pi r\ell$
Cylinder	$V = \pi r^2 h$	$LSA = 2\pi r h$
Prism	$V = Bh$	depends on base shape
Pyramid	$V = \frac{1}{3}Bh$	depends on base shape

In the formula chart, SA and TSA is the total surface area and LSA is the lateral surface area (the area along the sides only). To get the total surface area, add the area of the base(s) to the lateral surface area. The B in the formulas for prisms and pyramids is the area of the base. This allows the base to be square, rectangular, triangular, or any other shape!

EXAMPLE:

Three vertices of a rectangular box are given. Find the volume of the box.



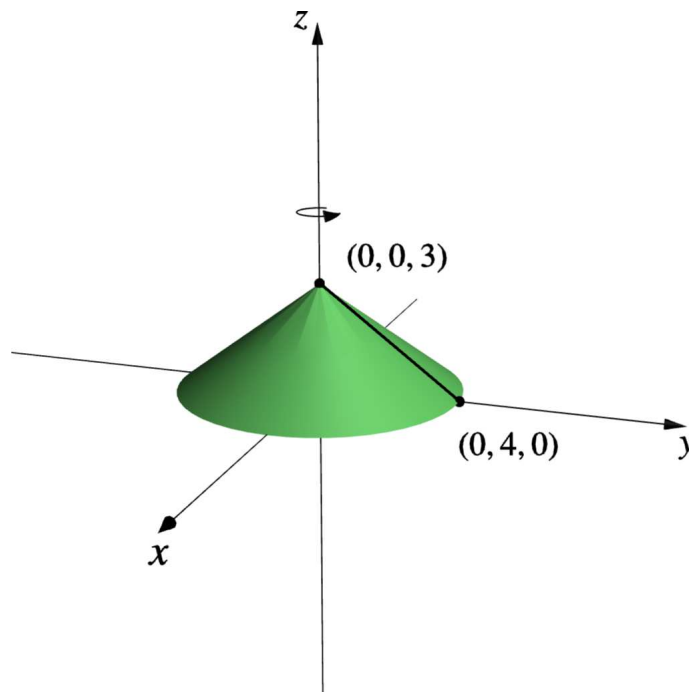
The box sits in the xy -plane, so to find the length and the width, we just calculate the changes in the x - and y -coordinates for the two vertices in the base. The change in x is $3 - 1 = 2$ and the change in y is $5 - 1 = 4$. Thus, the length is 2 and the width is 4. Next, to find the height, notice that the point $(1, 5, 4)$ is directly above the vertex $(1, 5, 0)$. This means that the height is the change in the z -coordinates and is $4 - 0 = 4$. The volume of the box is $V = \ell wh = 2(4)(4) = 32$.

Pappus's Theorems

One of Pappus's Theorem gives the surface area of a surface of revolution that is generated by revolving a curve around an axis. The formula for surface area is $SA = 2\pi s\bar{x}$, where s is the length of the curve and \bar{x} is distance the geometric centroid of the curve is from the axis of revolution.

EXAMPLE:

The segment shown is revolved around the z -axis. Find the surface area of the cone formed.



To find the surface area, we first need to find the centroid of the curve. Since the curve is a segment, the centroid is the midpoint. The midpoint is $(0, 2, 1.5)$. Next, we need to find the distance that the centroid is from the axis of revolution. Since the axis of revolution is the z -axis, the distance is 2, which is the length of the segment perpendicular to the axis to the centroid.

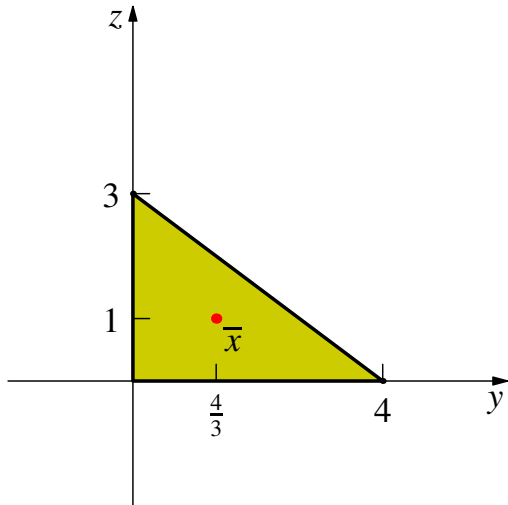
Next, we need to the length of the curve. Since the curve is a segment, you can use the distance formula. Alternatively, recognize that the segment along with the portions of the y - and z -axes form a right triangle with the right angle at the origin. The right triangle is a 3-4-5 triangle and thus, the length of the segment is 5.

Putting it all together, the surface area is $SA = 2\pi s\bar{x} = 2\pi(5)(2) = 20\pi$.

Another theorem from Pappus is for finding the volume of the solid of revolution formed by revolving a figure around an axis. To apply this theorem, you need the area of the figure being revolved A and the centroid of that figure \bar{X} . The volume is $V = 2\pi A\bar{X}$. Notice that \bar{x} from the surface area formula and \bar{X} from the volume formula are different centroids.

EXAMPLE:

Find the volume of the cone formed by revolving the triangle formed by the origin and the points $(0, 4, 0)$ and $(0, 0, 3)$ around the z -axis, as in the previous problem.



The graph shows the cross-section, which is a triangle, in the yz -plane. First, let's find the centroid of the triangle. The y -coordinate of the centroid can be found by adding all three y -coordinates of the vertices and dividing by 3. The z -coordinate of the centroid can be found the same way using the z -coordinates of the vertices.

$$G_y = \frac{0 + 4 + 0}{3} = \frac{4}{3} \quad \text{and} \quad G_z = \frac{0 + 0 + 3}{3} = 1$$

Since the axis of revolution is the z -axis, the distance the centroid is from the axis is the y -coordinate: $\frac{4}{3}$.

Next, we need the area of the triangle: $A = \frac{1}{2}bh = \frac{1}{2}(4)(3) = 6$.

The volume of the solid of revolution is $V = 2\pi A\bar{X} = 2\pi(6)\left(\frac{4}{3}\right) = 16\pi$.

Practice Questions

- Find the distance between the points $(5, 3, -1)$ and $(2, 9, 4)$.
- Find the midpoint of the segment whose endpoints are $(1, -1, 2)$ and $(-7, -5, -4)$.
- Find the center and radius of the sphere whose equation is $(x + 3)^2 + (y - 8)^2 + (z - 5)^2 = 81$.
- Find the center and radius of the sphere whose equation is $x^2 + y^2 + z^2 + 8x + 12y - 2z = 47$.
- What is the largest value of y such that the point $(8, y, 10)$ is 7 units away from the point $(6, 9, 4)$?
- Find the intercepts of the plane $8x - 3y + 2z = 24$.
- The plane $Ax + By + Cz = 60$ has an x -intercept of -5 , a y -intercept of 6, and a z -intercept of 4. Find A , B , and C .
- Find the volume of the pyramid enclosed in the first octant by the origin and the intercepts of the plane $x + 2y + z = 6$.
- Find the point of intersection of the planes $x + y + z = 12$, $2x + y - z = 20$, and $-x - y + 4z = -17$.
- Find the surface area of the solid of revolution formed by revolving the segment whose endpoints are $(2, 0, 0)$ and $(1, 0, 1)$ around the z -axis.
- Find the volume of the solid of revolution formed by revolving the triangle whose vertices are $(2, 0, 0)$, $(1, 0, 1)$, and $(1, 0, 0)$ around the z -axis.
- The figure shown in the yz -plane is revolved around the z -axis. Find the volume of the resulting solid.

